JOURNAL OF APPROXIMATION THEORY 14, 152-159 (1975)

The Approximate Solution of Volterra Integral Equations

A. G. PETSOULAS

Department of Mathematics, University of Athens, Athens, Greece

Communicated by Richard S. Varga

Huffstutler and Stein and recently Bacopoulos and Kartsatos have dealt with the problem of best approximation by polynomials of the solutions of nonlinear differential equations. The purpose of the present paper is to generalize their results and to show that they can be established under a weaker set of conditions.

We consider the best approximation by polynomials of the solutions on [0, 1] of the Volterra integral equation

$$L(x) = x(t) + \int_0^t f(t, s, x(s))\, ds = h(t) \qquad (1)$$

as it is stated in [1] and [2]. The functions f, h are defined and continuous on $[0, 1] \times [0, 1] \times \mathbb{R}$, $[0, 1]$, respectively. Suppose that there exists a unique solution $x(t)$ of (1) defined on [0, 1]. On $C[0, 1]$ we consider the norm

$$\| \varphi \| = \sup_{t \in [0,1]} | \varphi(t) |, \qquad \varphi \in C[0, 1].$$

Let Π_n be the set of all polynomials P_n of degree less than or equal to n which satisfy the condition $P_n(0) = h(0)$, and put $\mu_n = \inf_{P \in \Pi_n} \| L(x) - L(P) \|$, $n = 1, 2, \ldots$. We examine if there exist polynomials $P_n \in \Pi_n$ such that

$$\mu_n = \| L(x) - L(P_n) \| \qquad \text{and} \qquad \lim_{n \to \infty} P_n = x, \quad \text{uniformly on } [0, 1].$$

The results obtained in this paper contain as special cases those of [1] and [2]. It is also to be noted that some of the conditions in [2] are not necessary.

1. PRELIMINARIES

Let $I \subseteq \mathbb{R}$ be an interval with left endpoint zero. If x is a real continuous function defined on I, then the operator $Q : I \times x \to C[-1, 0]$ is defined by

$$(Q_t x)(\theta) = x(t(1 + \theta)), \quad \theta \in [-1, 0], \quad t \in I.$$

Let $g: I \times C[-1, 0] \to \mathbb{R}$ and $f: I \times C[-1, 0] \to \mathbb{R}$ be continuous functions. A hereditary differential equation is a relation of the form

$$\frac{d}{dt}(D(t)\, Q_t x) = f(t, Q_t x),$$

where

$$D(t)\,\varphi = \varphi(0) - g(t, \varphi),\ t \in I,\ \varphi \in C[-1, 0].$$

Suppose $U \subseteqq I \times C[-1, 0]$ is open and

$$S(t, \varphi, \psi, \mu, s) = \{\psi \in C[-1, 0] : (t, \psi) \in U, \| \psi - \varphi \| \leqslant \mu,$$
$$\psi(\theta) = \varphi(\theta), \theta \in [-1, -s]\},$$

$(t, \varphi) \in U$. We say that a continuous function $g : U \to \mathbb{R}$ is nonatomic at zero if for every $(t, \varphi) \in U$ there exist $s_0 = s_0(t, \varphi) > 0$, $\mu_0 = \mu_0(t, \varphi)$ continuous and $p(t, \varphi, \mu, s)$ nondecreasing in μ, s and continuous such that

$$p(t, \varphi, \mu, s) < 1 \qquad \text{and} \qquad |g(t, \psi) - g(t, \varphi)| \leqslant p(t, \varphi, \mu, s)\| \psi - \varphi \|$$

for every $(t, \psi) \in U$, $\psi \in S(t, \varphi, \mu, s)$, $s \in [0, s_0]$, $\mu \in [0, \mu_0]$.

2. Main Results

THEOREM 1. *We consider a solution $x(l)$ of the* Eq. (1) *on* $[0, 1]$. *If the function*

$$g(t, \varphi) = t \int_{-1}^{0} f(t, t(1 + \theta), \varphi(\theta))\, d\theta, \qquad t \in [0, 1], \quad \varphi \in C[-1, 0]$$

is nonatomic at zero, then there exist an integer $n_0 \geqslant 0$ and $P_n \in \Pi_n$ such that

$$\| L(x) - L(P_n)\| = \inf_{P \in \Pi_n} \| L(x) - L(P)\|, \qquad n \geqslant n_0$$

and $\lim_{n\to\infty} P_n = x$, *uniformly on* $[0, 1]$.

The proof of this Theorem requires the following lemmas.

LEMMA 1. *We consider the integral equations*

$$(V_n) \qquad x(t) = \int_0^t f_n(t, s, x(s))\, ds + h_n(t), \qquad n = 1, 2, \ldots,$$

where $f_n : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$, $h_n : [0, 1] \to \mathbb{R}$ are continuous functions *such that* $\lim_{n\to\infty} f_n = f_0$, $\lim_{n\to\infty} h_n = h_0$, *uniformly on* $[0, 1] \times [0, 1] \times \mathbb{R}$, $[0, 1]$, *respectively. If*

$$g_n(t, \varphi) = t \int_{-1}^{0} f_n(t, t(1 + \theta), \varphi(\theta))\, d\theta + h_n(t),$$

$$t \in [0, 1], \quad \varphi \in C[-1, 0], \quad n = 0, 1, \ldots$$

and g_0 *is nonatomic at zero, then there exist an integer* $n_0 \geqslant 0$ *and solutions* x_n *of* (V_n) *on* $[0, 1]$, $n \geqslant n_0$ *such that* $\lim_{n\to\infty} x_n = x_0$, *uniformly on* $[0, 1]$.

Proof. From what it is stated in [4, p. 67], the nonatomic property at zero of g_0 implies uniqueness of the solutions of (V_0) and the hereditary equations

$$(H_n) \qquad \frac{d}{dt}(x(t) - g_n(t, Q_t x)) = 0$$

$$x(0) = h_n(0), \qquad n = 0, 1, \ldots$$

are equivalent to the integral equations

$$(V_n) \qquad x(t) = \int_0^t f_n(t, s, x(s))\, ds + h_n(t), \qquad n = 0, 1, \ldots .$$

Thus, the existence of solutions x_n of (V_n) with $\lim x_n = x_0$, uniformly on $[0, 1]$ follows from the existence of solutions x_n of (H_n) with $\lim_{n\to\infty} x_n = x_0$, uniformly on $[0, 1]$.

On the other hand, by Theorem 6.2 in [4], there exists a sequence x_n of solutions of (H_n) which converges uniformly on $[0, 1]$ if

(i) $\lim_{n\to\infty} g_n = g_0$, uniformly on closed and bounded subsets of $[0, 1] \times C[-1, 0]$;

(ii) g_n, $n = 0, 1, \ldots$ are compact;

(iii) g_0 is nonatomic at zero; and

(iv) g_0 is uniformly continuous on closed and bounded subsets of $[0, 1] \times C[-1, 0]$.

It is clear that all these conditions are satisfied and this completes the proof of the lemma.

LEMMA 2. *Let* x *be a solution of the* Eq. (1) *on* $[0, 1]$. *If the function*

$$g(t, \varphi) = t \int_{-1}^{0} f(t, t(1 + \theta), \varphi(\theta))\, d\theta, \qquad t \in [0, 1], \quad \varphi \in C[-1, 0]$$

is nonatomic at zero and there exist $Q_n \in \Pi_n$ *such that*

$$\| L(x) - L(Q_n)\| \leqslant \mu_n + \epsilon_n , \qquad n = 1, 2, \ldots,$$

where

$$\epsilon_n \geqslant 0, \qquad \lim_{n\to\infty} \epsilon_n = 0,$$

then Q_n *converges uniformly to the solution* x *on* $[0, 1]$.

Proof. We prove first that $\lim_{n\to\infty} L(Q_n) = L(x)$, uniformly on $[0, 1]$. In fact, according to Weierstrass theorem there exist polynomials $S_n \in \Pi_n$, $n = 1, 2,\ldots$ such that

$$\lim_{n\to\infty} S_n = x, \qquad \text{uniformly on } [0, 1]. \tag{2}$$

On the other hand we have

$$\begin{aligned} &\| L(x) - L(Q_n)\| \\ &\quad \leqslant \mu_n + \epsilon_n \leqslant \| L(x) - L(S_n)\| + \epsilon_n \\ &\quad = \left\| x(t) + \int_0^t f(t, s, x(s))\, ds - S_n(t) - \int_0^t f(t, s, S_n(s))\, ds \right\| + \epsilon_n \\ &\quad \leqslant \| x - S_n \| + \left\| \int_0^t f(t, s, x(s))\, ds - \int_0^t f(t, s, S_n(s))\, ds \right\| + \epsilon_n . \end{aligned} \tag{3}$$

Thus, by (2) and (3), we obtain

$$\lim_{n\to\infty} \| L(x) - L(Q_n)\| = 0. \tag{4}$$

We show next that $\lim_{n\to\infty} Q_n = x$, uniformly on $[0, 1]$. If we put

$w_n(t) = x(t) - Q_n(t)$ and $k_n(t) = L(x(t)) - L(Q_n(t))$, $t \in [0, 1]$, $n = 1, 2,\ldots;$

then

$$\begin{aligned} k_n(t) &= L(x(t)) - L(Q_n(t)) = L(x(t)) - L(x(t) - w_n(t)) \\ &= x(t) + \int_0^t f(t, s, x(s))\, ds \\ &\qquad - \left(x(t) - w_n(t) + \int_0^t f(t, s, x(s) - w_n(s))\, ds\right) \\ &= w_n(t) + \int_0^t f(t, s, x(s))\, ds - \int_0^t f(t, s, x(s) - w_n(s))\, ds. \end{aligned}$$

Therefore, the functions w_n are solutions of the equations

$$(V_n) \quad w(t) = \int_0^t f(t, s, x(s) - w(s))\, ds - \int_0^t f(t, s, x(s))\, ds + k_n(t),$$
$$n = 1, 2, \ldots .$$

From (4) we obtain $\lim_{n\to\infty} k_n = 0$, uniformly on $[0, 1]$. On the other hand the solution of the equation

$$(V_0) \qquad w(t) = \int_0^t f(t, s, x(s) - w(s))\, ds - \int_0^t f(t, s, x(s))\, ds$$

is $w = 0$ on $[0, 1]$. Thus, by Lemma 1, the sequence w_n of the solutions of (V_n) converges uniformly to the solution $w = 0$ of (V_0). Hence, $\lim_{n\to\infty} Q_n = x$, uniformly on $[0, 1]$.

LEMMA 3. *If* $x(t)$, $t \in [0, 1]$ *is a solution of* (1) *and* $\min_{P\in\Pi_k} \| L(x) - L(P)\|$ *does not exist, then there exists an unbounded sequence* $Q_{k,n} \in \Pi_k$, $n = 1, 2,\ldots$ *such that* $\lim_{n\to\infty} \| L(x) - L(Q_{k,n})\| = \mu_k$.

Proof. Since $\mu_k = \inf_{P\in\Pi_k} \| L(x) - L(P)\|$, there exists $Q_{k,n} \in \Pi_k$, $n = 1, 2,\ldots$ such that

$$\lim_{n\to\infty} \| L(x) - L(Q_{k,n})\| = \mu_k . \tag{5}$$

The sequence $Q_{k,n}$, $n = 1, 2,\ldots$ is unbounded because, in contrary, we have $Q_{k,n}$, $n = 1, 2,\ldots$ bounded and consequently $\int_0^t f(t, s, Q_{k,n}(s))\, ds$, $n = 1, 2,\ldots$ is equicontinuous. Also, by (5), the sequence $L(Q_{k,n})$, $n = 1, 2,\ldots$ is equicontinuous on $[0, 1]$. From these and since

$$L(Q_{k,n}) = Q_{k,n}(t) + \int_0^t f(t, s, Q_{k,n}(s))\, ds$$

we have that $Q_{k,n}$, $n = 1, 2,\ldots$ is equicontinuous on $[0, 1]$ and consequently there exists a subsequence $Q_{k,\lambda n}$ such that $\lim_{n\to\infty} Q_{k,\lambda n} = P_k \in \Pi_k$.

Thus, by (5),

$$\mu_k = \lim_{n\to\infty} \| L(x) - L(Q_{k,\lambda n})\| = \| L(x) - L(P_k)\| ,$$

which is a contradition.

Proof of Theorem 1. If the first result of the Theorem does not hold, then there exists an increasing sequence λn of integers such that

$$\min_{P \in \Pi_{\lambda n}} \| L(x) - L(P) \|$$

does not exist. Thus, by Lemma 3, for every λn there exist $Q_{\lambda n} \in \Pi_{\lambda n}$, which satisfy the relations

$$\| L(x) - L(Q_{\lambda n}) \| \leqslant \mu_{\lambda n} + (1/\lambda n) \tag{6}$$

$$\| Q_{\lambda n} \| > \lambda n, \qquad n = 1, 2, \ldots. \tag{7}$$

From (6) and Lemma 2 we obtain $\lim_{n\to\infty} Q_{\lambda n} = x$, uniformly on [0, 1], which is a contradiction to (7).

Now, since there exist an integer $n_0 \geqslant 0$ and $P_n \in \Pi_n$ such that

$$\| L(x) - L(P_n) \| = \min_{P \in \Pi_n} \| L(x) - L(P) \| = \mu_n, \qquad n \geqslant n_0,$$

by Lemma 2, we have

$$\lim_{n\to\infty} P_n = x, \qquad \text{uniformly on } [0, 1].$$

COROLLARY 1. *Let the function f in* (1) *be such that*

$$| f(t, s, u) - f(t, s, v) | \leqslant A \sum_{k=1}^{m} | u^k - v^k |,$$

$$(t, s, u, v) \in [0, 1] \times [0, 1] \times \mathbb{R}$$

where A is a positive constant. Then there exist an integer $n \geqslant n_0$ *and* $P_n \in \Pi_n$ *such that*

$$\| L(x) - L(P_n) \| = \min_{P \in \Pi_n} \| L(x) - L(P) \|, \qquad n \geqslant n_0$$

and

$$\lim_{n\to\infty} P_n = x, \qquad \textit{uniformly on } [0, 1].$$

Proof. The function

$$g(t, \varphi) \equiv t \int_{-1}^{0} f(t, t(1 + \theta), \varphi(\theta))\, d\theta, \qquad t \in [0, 1], \quad \varphi \in C[-1, 0]$$

is nonatomic at zero since

$$
\begin{aligned}
&| g(t, \psi) - g(t, \varphi)| \\
&\quad = \left| t \int_{-1}^{0} (f(t, t(1 + \theta), \psi(\theta)) - f(t, t(1 + \theta), \varphi(\theta)))\, d\theta \right| \\
&\quad \leqslant \int_{-s}^{0} \sum_{k=1}^{m} |(\psi(\theta))^k - (\varphi(\theta))^k|\, d\theta \\
&\quad \leqslant \| \psi - \varphi \| s \sum_{k=1}^{m} (| \psi(\theta)|^{k-1} + | \psi(\theta)|^{k-2} | \varphi(\theta)| + \cdots + | \varphi(\theta)|^{k-1}) \\
&\quad \leqslant \| \psi - \varphi \| s \sum_{k=1}^{m} ((\| \varphi \| + \mu)^{k-1} + \cdots + \| \varphi \|^{k-1})
\end{aligned}
$$

for any $(t, \varphi) \in [0, 1] \times C[-1, 0]$ and $\psi \in S(t, \varphi, \mu, s)$. Hence, this Corollary follows from Theorem 1.

By the same idea as in the proof of Theorem 1 we can prove the following theorem.

THEOREM 2. *Let $x(t)$, $t \in [0, 1]$ be a solution of the initial value problem*

$$
(\Pi) \qquad \begin{aligned} M(x) &\equiv x' + F(t, x) = G(t) \\ x(0) &= a, \end{aligned}
$$

where $F : [0, 1] \times \mathbb{R} \to \mathbb{R}$, $G : [0, 1] \to \mathbb{R}$ are continuous functions and

$$
a \in \mathbb{R}. \text{ If } g(t, \varphi) = t \int_{-1}^{0} F(t(1 + \theta), \varphi(\theta))\, d\theta, \quad t \in [0, 1], \quad \varphi \in C[-1, 0]
$$

is nonatomic at zero, then there exist an integer $n_0 \geqslant 0$ and $P_n \in \Pi_n{}^*$ ($\Pi_n{}^*$ is the set of all polynomials of degree less than or equal to n with $\Pi_n{}^*(0) = a$) such that

$$
\begin{aligned}
&\left(\int_0^1 | M(x(t)) - M(P_n(t))|^p\, dt\right)^{1/p} \\
&\quad = \| M(x) - M(P_n)\|_p = \min_{P \in \Pi_n{}^*} \| M(x) - M(P)\|_p , \quad n \geqslant n_0 \quad (p \geqslant 1)
\end{aligned}
$$

and

$$
\lim_{n \to \infty} P_n = x, \qquad \textit{uniformly on } [0, 1].
$$

COROLLARY 2. *Let* $x(t)$, $t \in [0, 1]$ *be a solution of the initial value problem* (Π). *If the function* F *satisfies the condition*

$$|F(t, u) - F(t, v)| \leqslant A \sum_{k=1}^{m} |u^k - v^k|, \qquad (t, u, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R},$$

where A *is a positive constant, then the result of Theorem* 2 *holds.*

Remark. From the above corollary it is obvious that the theorems in [1] as well as the theorem in [2] are special cases of Theorem 2. Also, the conditions on the constant A in [2] can be omitted.

REFERENCES

1. R. G. HUFFSTUTLER AND F. MAX STEIN, The approximate solution of certain nonlinear differential equations, *Proc. Amer. Math. Soc.* **19** (1968), 998–1002.
2. A. BACOPOULOS AND A. G. KARTSATOS, On polynomials approximating the solutions of nonlinear differential equations, *Pacific J. Math.* **40** (1972), 1–5.
3. W. A. COPPEL, "Stability and asymptotic behavior of differential equations," Heath, Boston, MA, 1965.
4. J. K. HALE AND M. A. CRUZ, Existence, uniqueness, and continuous depence for hereditary systems, *Ann. Mat. Pura Appl.* (4) **85** (1970), 63–81.